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The steady temperature distributions with different types of nonlinearities

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ABSTRACT

The nonlinear two-point boundary value problems are solved and the steady temperature distributions in a rod are found by considering different types of the nonlinear parts of the problems, particularly in the polynomial, exponential and trigonometric forms. In this paper, with the aid of some transformations the variational iteration method's scheme is reproduced for the nonlinear problems including two-point boundary value problems. The illustrative related problems are solved by means of the method scheme.

1. Introduction

The steady state temperature distributions in a rod of length L are considered by Wang [1] as the two point boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k(T)\frac{\mathrm{d}T}{\mathrm{d}x}\right) = f(x,T), \quad 0 \le x \le L, \qquad (1)$$
$$\text{subject to boundary conditions } T(0) = \alpha \quad \text{and} \quad T(L) = \beta$$

where the ends of the rod are held at the given constant temperatures α and β, the function $k(T)$ that depends on the temperature distribution, represents the thermal conductivity and it change with temperatures. Generally the thermal conductivity functions are taken to be linear functions in many engineering applications, the problem is getting more complicated while the functions $k(T)$, $f(x,T)$ are chosen nonlinear functions with respect to T, particularly such as exponential types and trigonometric types in our investigations. For the problem (1), the approximate solutions by means of the numerical method are able to use mostly for solving those heat transfer problems with both homogeneous and non-homogeneous boundary conditions. In [2], Wang used a transformation that transforms (1) into a feasible second order boundary value problem in order to apply Numerov's method developed by Agarwal and Wang. So, the second order two-point boundary value problem (1) in the general form is considered and its approximate solutions are sought to find the steady temperature distributions in a rod of length L. For this purpose, it is chosen the variational iteration method to calculate theirs exact solutions. The advantage of the chosen method is its celerity in reaching to approximate solutions with the rapid accuracies. After transforming problem to a suitable second order differential equation, it can be solved by various methods which can give the numerical solutions such as numerov method [1,2], Adomian decomposition method, differential transform method [3] etc. In this paper, choosing $k(T)$ as a linear or a nonlinear function of T, the approximate solutions of (1) are obtained by variational iteration method. The variational iteration method [3–17] are employed by many authors to solve the famous ordinary and partial differential equations, the mathematical models in various fields. The problem (1) would become different types that are required to prepare it to a suitable differential equation form in order to solve it by the variational iteration method.

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2. Inversion principle

In order to derive the suitable form of differential equation (1) for applying the variational iteration method, (1) will be transformed to a proper problem by making use of the inversion principle. Let F be a continuous function on $\mathbb{R}$ with inverse F^{-1} defined by

$$F^{-1}(v) = \{T : F(T) = v, v_0 < v < v_1\}.$$

The interval I is a subset of $\mathbb{R}$ including the end points α and β of the problem. The function $k(T)$ is a continuous function of T in I and there exist positive constants k_0 and k_1 such that $k_0 \leq k(T) \leq k_1$ in I [1]. Then the transformation F is

$$v = F(T) = \int_{\rho}^{T} k(s)\mathrm{d}s \quad \text{and} \quad \forall u \in I. \tag{2}$$

Since F is a strictly monotone increasing function of T in I, F^{-1} exists. Using inverse transformation $T = F^{-1}(v)$, problem (1) is transformed into the following nonlinear second order boundary value problem:

$$\begin{aligned} &\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = f(x, F^{-1}(v)), \quad 0 \leq x \leq L, \\ &\text{subject to boundary conditions } v(0) = F(\alpha) \quad \text{and} \quad v(L) = F(\beta). \end{aligned} \tag{3}$$

If the above problem is solved for $v = F(T)$, then problem (1) will be solved for T by the above described inverse transformation [1].

3. The application process of the variational iteration method to the transformed problems

The variational iteration method is used to solve a wide class of the differential equations [4–9] and its methodology is also given by many authors from the ordinary differential equations to the partial differential equations in such papers [3,10–16]. The suitable process for the problem (3) is already presented and constructed in [17]. Briefly, the function $f(x, F^{-1}(v))$ can be taken as the function $g(x, v)$ and the problem becomes

$$G\left(x, v, v''\right) = 0 \tag{4}$$

where x is the independent variable, v is the unknown function and G has both linear part Lv and nonlinear part Nv. The correction functional of (4) is

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\, G\left(\xi, v_n, v_n''\right) \mathrm{d}\xi$$

and

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\, \left(v_n''(\xi) - g(\xi, v_n)\right) \mathrm{d}\xi. \tag{5}$$

If $g(x, v)$ is separated of its variables such that ψ is a function of x and ϕ is a function of v, (5) can be written as either

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\, \left(v_n''(\xi) - \psi(\xi)\phi(v_n)\right) \mathrm{d}\xi. \tag{6}$$

or

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\, \left(v_n''(\xi) - \psi(\xi) - \phi(v_n)\right) \mathrm{d}\xi. \tag{7}$$

Especially, if the function $\phi(v)$ is a continuous differentiable nonlinear function in the interval that we are interested, that is, polynomial function degree of ≥ 2 as it is considered in [17], exponential function or trigonometric function with respect to v; then making stationary (6) and (7) and noticing that $\delta \tilde{v}_n$ is restricted variation, (6) and (7) can be written as follows:

$$\delta v_{n+1}(x) = \delta v_n(x) + \delta \int_0^x \lambda(\xi)\, \left(v_n''(\xi) - \psi(\xi)\phi(v_n)\right) \mathrm{d}\xi. \tag{8}$$

$$\delta v_{n+1}(x) = \delta v_n(x) + \delta \int_0^x \lambda(\xi)\, \left(v_n''(\xi) - \psi(\xi) - \phi(v_n)\right) \mathrm{d}\xi. \tag{9}$$

The Lagrange multiplier can be readily identified from above Eqs. (8) and (9).

The problem (3) is now investigated for the three well known nonlinearities. For these specific problems the Lagrange multipliers are found and the correction functional are constructed as follows.

Case 1. We first consider the condition that $\phi(v)$ can be a polynomial. It can be noted that here, the numerical solutions and analysis for this case are studied in [17]. Here the process is how we can find the Lagrange multiplier:

Corollary. *If the polynomial family $\phi_p(v_n)$ degree of p greater than one is considered in the iteration formula (6) and (7) such that the coefficients are $\beta_p \neq 0$ and $\beta_k \in \mathbb{R}$ for $k = 0, 1, 2, \dots, p$ as*

$$\phi_p(v_n) = \beta_0 + \beta_1 v_n + \cdots + \beta_p v_n^p,$$

then the constant term and the nonlinear terms have to be zero and there remains

$$\delta\phi_p(\tilde{v}_n) = \beta_1 \delta v_n.$$

Its proof is immediate by the condition $\delta\tilde{v}_{n+1} = 0$ and neglecting the constant term

$$\begin{aligned}\delta\phi_p(v_n) &= \delta\beta_0 + \delta\beta_1 v_n + \delta\beta_2 \tilde{v}_n^2 + \cdots + \delta\beta_p \tilde{v}_n^p \qquad \text{for } p \geq 2.\\ &= \beta_1 \delta v_n,\end{aligned}$$

In the view of the corollary, the Eq. (8) can be rearranged and the Lagrange multiplier has to be compute in the following process

$$\delta v_{n+1}(x) = \delta v_n(x) + \lambda(\xi)\delta v_n'(\xi)|_{\xi=x} - \lambda'(\xi)\delta v_n(\xi)|_{\xi=x} + \int_0^x \{\lambda''(\xi)\, \delta v_n(\xi) - \lambda\psi(\xi)[\delta\beta_0 + \delta\beta_1 v_n(\xi)]\}\mathrm{d}\xi = 0$$

and the conditions of Lagrange Multiplier are

$$\begin{aligned}&\delta v_n(x) : 1 - \lambda'(\xi)|_{\xi=x} = 0\\ &\delta v_n'(\xi) : \lambda(\xi)|_{\xi=x} = 0\\ &\delta v_n(\xi) : \lambda''(\xi)|_{\xi=x} - \beta_1\psi(\xi)\lambda(\xi)|_{\xi=x} = 0.\end{aligned}$$

Case 2. Secondly, we consider the condition that $\phi(v)$ can be an exponential function:

Corollary. *If the exponential function is considered as*

$$\phi(v_n) = a_0 \mathrm{e}^{a_1 v_n}$$

where the coefficients are $a_i \neq 0$ and $a_i \in \mathbb{R}$ for $i = 0, 1$ then $\delta\phi(v_n) = a_0 a_1 \delta v_n$.

For instance, assuming $\psi(x) = x$ and $\phi(v) = \mathrm{e}^v - v$, we have the following nonlinear problem

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = x(\mathrm{e}^v - v) \tag{10}$$

with the boundary conditions $v(0) = F(\alpha) = \gamma$ and $v(L) = F(\beta) = \mu$. By the above corollary, the Eq. (8) can be rearranged and the Lagrange multiplier is found as $\lambda(\xi) = \xi - x$ so that the correction functional is

$$v_{n+1}(x) = v_n(x) + \int_0^x (\xi - x)\,\left(v_n''(\xi) - \xi\left(\mathrm{e}^{v_n(\xi)} - v_n(\xi)\right)\right)\mathrm{d}\xi.$$

It is convenient to take $v_0(x) = ax + b$ then

$$v_1(x) = b + ax + \frac{1}{6}x^3 + \frac{1}{12}b^2x^3 + \frac{1}{12}abx^4 + \frac{1}{40}a^2x^5$$

and from the left boundary condition, the coefficient of starting function, b is equal to γ and from the right boundary condition $a = -\frac{1}{3L^4}\left(5L^3\gamma - 60 + \sqrt{-5L^6\gamma^2 + 240L^3\gamma + 3600 + 360L^3\mu - 60L^6}\right)$. The first approximate solution of (10) is as

$$\begin{aligned}v_1(x) = \gamma &- \frac{1}{3L^4}\left(5L^3\gamma - 60 + \sqrt{-5L^6\gamma^2 + 240L^3\gamma + 3600 + 360L^3\mu - 60L^6}\right)x\\ &+ \frac{1}{6}x^3 + \frac{1}{12}\gamma^2x^3 - \frac{1}{36L^4}\left(5L^3\gamma - 60 + \sqrt{-5L^6\gamma^2 + 240L^3\gamma + 3600 + 360L^3\mu - 60L^6}\right)\gamma x^4\\ &+ \frac{1}{40}\left(-\frac{1}{3L^4}\left(5L^3\gamma - 60 + \sqrt{-5L^6\gamma^2 + 240L^3\gamma + 3600 + 360L^3\mu - 60L^6}\right)\right)^2 x^5.\end{aligned}$$

In the second iteration, $v_2(x)$ can be found by using $v_1(x)$, and proceeding this process the ith approximate solution $v_i(x)$ for $2 \leq i \leq n$ can also be generated.

Case 3. Finally, we consider the condition that $\phi(v)$ can be a trigonometric function.

Corollary. *If the trigonometric function is considered as*

$$\phi(v_n) = a_0 \sin(\omega v_n) + a_1 \cos(\omega v_n)$$

where the coefficients are $a_i \in [1, \infty)$ *for* $i = 0, 1$ *and* $\omega \in (0, 1]$ *then* $\delta\phi(v_n) = a_0\omega\delta v_n$.

For its proof, let

$$\sin(\omega v_n) = \sum_{k=0}^{\infty}(-1)^k \frac{(\omega v_n)^{2k+1}}{(2k+1)!} \quad \textit{and} \quad \cos(\omega v_n) = \sum_{k=0}^{\infty}(-1)^k \frac{(\omega v_n)^{2k}}{2k!}$$

for $-\infty < \omega v_n < \infty$. *Then*

$$\phi(v_n) = a_0 \sin(\omega v_n) + a_1 \cos(\omega v_n) = a_0 \sum_{k=0}^{\infty}(-1)^k \frac{(\omega v_n)^{2k+1}}{(2k+1)!} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{(\omega v_n)^{2k}}{2k!}$$

$$\phi(v_n) = a_0 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k+1} v_n^{2k+1}}{(2k+1)!} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k} v_n^{2k}}{2k!}$$

$$\delta\phi(v_n) \equiv \delta a_0 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k+1} v_n^{2k+1}}{(2k+1)!} + \delta a_1 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k} v_n^{2k}}{2k!}$$

$$\delta\phi(v_n) \equiv a_0 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k+1} \delta v_n^{2k+1}}{(2k+1)!} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k} \delta v_n^{2k}}{2k!}$$

$$\delta\phi(v_n) \equiv a_0\omega\delta v_n + \sum_{k=1}^{\infty}(-1)^k \frac{\omega^{2k+1} \delta \tilde{v}_n^{2k+1}}{(2k+1)!} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k} \delta \tilde{v}_n^{2k}}{2k!}$$

$$\delta\phi(v_n) \equiv a_0\omega\delta v_n + \sum_{k=1}^{\infty}(-1)^k \frac{\omega^{2k+1} \delta \tilde{v}_n^{2k+1}}{(2k+1)!} + a_1 \sum_{k=0}^{\infty}(-1)^k \frac{\omega^{2k} \delta \tilde{v}_n^{2k}}{2k!}.$$

By the restricted variation $\delta\tilde{v}_n = 0$, *the last expression is identically become*

$$\delta\phi(v_n) \equiv a_0\omega\delta v_n. \quad \square \tag{11}$$

Considering the assumption (11), the Eq. (8) can be rearranged as

$$\delta v_{n+1}(x) = \delta v_n(x) + \lambda(\xi)\delta v_n'(\xi)|_{\xi=x} - \lambda'(\xi)\delta v_n(\xi)|_{\xi=x} + \int_0^x \left(\lambda''(\xi) - a_0\omega\lambda(\xi)\right)\delta v_n(\xi)\mathrm{d}\xi = 0$$

then the Lagrange multiplier is $\lambda(\xi) = \frac{\sqrt{a_0\omega}}{a_0\omega}\sinh(\xi - x)$.

For instance, assuming $\psi(x) = 1$ and $\phi(v_n) = a_0 \sin(\omega v_n) + a_1 \cos(\omega v_n)$, we have the following nonlinear problem

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = a_0 \sin(\omega v) + a_1 \cos(\omega v) \tag{12}$$
with the boundary conditions $v(0) = F(\alpha) = \gamma$ and $v(L) = F(\beta) = \mu$.

The correction functional is

$$v_{n+1}(x) = v_n(x) + \int_0^x \frac{\sqrt{a_0\omega}}{a_0\omega}\sinh(\xi - x)\left(v_n''(\xi) - a_0\omega v_n(\xi) + \frac{a_0}{6}\omega^3 v_n^3(\xi) - a_1 + \frac{a_1}{2}\omega^2 v_n^2(\xi)\right)\mathrm{d}\xi.$$

Taking the starting function as $v_0(x) = ax + b$ yields

$$\begin{aligned} v_1(x) = \frac{1}{12}\frac{1}{\sqrt{a_0 w}}[&12a_1w^2a^2 - 12a_1 - 12a_0wb + 2a_0w^3b^3 + 6a_1w^2b^2 + 12a_0w^3a^3x + 12b\sqrt{a_0w} - 6w^2aa_1be^x \\ &+ 12ax\sqrt{a_0w} + 12a_0w^3a^2b + 12w^2aa_1xb + 6a_1e^{(-x)} - 6a_0w^3a^2be^{(-x)} + 6w^2aa_1be^{(-x)} \\ &- 3w^3aa_0b^2e^x - 6a_0w^3a^3e^x - a_0w^3b^3e^x - 6a_1w^2a^2e^x - 3a_1w^2b^2e^x + 6a_0wae^x \\ &+ 6a_0wbe^x - 6a_0w^3a^2be^x - 12a_0wax + 6a_1w^2a^2x^2 + 2a_0w^3a^3x^3 + 6w^3aa_0xb^2 + 6a_0w^3a^2x^2b \\ &+ 6a_1e^x - 6a_0wae^{(-x)} + 3w^3aa_0b^2e^{(-x)} + 6a_0wbe^{(-x)} \\ &+ 6a_0w^3a^3e^{(-x)} - a_0w^3b^3e^{(-x)} - 6a_1w^2a^2e^{(-x)} - 3a_1w^2b^2e^{(-x)}] \end{aligned} \tag{13}$$

is obtained and from the left boundary condition, the coefficient b of starting function is equal to γ and from the right boundary condition, a has to verify

$$\mu = \frac{1}{12}\frac{1}{\sqrt{a_0 w}}[18a_1 w^2 a^2 - 12a_0 wa + 14a_0 w^3 a^3 - 12a_1 - 12a_0 wb + 2a_0 w^3 b^3 + 6a_1 w^2 b^2 + 12b\sqrt{a_0 w} + 18a_0 w^3 a^2 b + 12w^2 aa_1 b + 6w^3 aa_0 b^2 - 6a_0 w^3 a^3 e - a_0 w^3 b^3 e - 6a_1 w^2 a^2 e - 3a_1 w^2 b^2 e + 6a_0 wae + 6a_0 wbe + 6a_1 e - 3w^3 aa_0 b^2 e - 6a_0 w^3 a^2 be - 6w^2 aa_1 be + 6a_1 \mathrm{e}^{(-1)} + 12a\sqrt{a_0 w} - 6a_0 wa\mathrm{e}^{(-1)} + 6a_0 wb\mathrm{e}^{(-1)} + 6a_0 w^3 a^3 \mathrm{e}^{(-1)} - a_0 w^3 b^3 \mathrm{e}^{(-1)} - 6a_1 w^2 a^2 \mathrm{e}^{(-1)} - 3a_1 w^2 b^2 \mathrm{e}^{(-1)} - 6a_0 w^3 a^2 b\mathrm{e}^{(-1)} + 6w^2 aa_1 b\mathrm{e}^{(-1)} + 3w^3 aa_0 b^2 \mathrm{e}^{(-1)}].$$

Therefore the approximate solution of (12) is (13) with above assumptions.

Up to now, for having the approximate solution of (3), the method is applied to the nonlinear second order boundary value problem (3), hereafter it is necessary to determine the transformation F by choosing the thermal conductivity function $k(T)$ as mentioned previous section. Three cases are studied for the thermal conductivity function as follows.

3.1. $k(T)$ polynomial function case [17]

If it is a polynomial

$$k(T) = a_2 T^2 + a_1 T + a_0 \tag{14}$$

where the coefficients are non-negative and $a_0 \neq 0$, then by using (2) and assuming $\rho = 0$

$$v = F(T) = \int_0^T (a_2 s^2 + a_1 s + a_0)\mathrm{d}s = \frac{a_2}{3}T^3 + \frac{a_1}{2}T^2 + a_0 T \quad \text{and} \quad \forall T \in I$$

is the transformation F whose inverse F^{-1} corresponds to a function of v

$$T = -\frac{a_1}{2a_2} + \frac{-9a_1^2 + 36a_0 a_2}{3\sqrt[3]{4}a_2\left(54a_1^3 - 324a_0 a_1 a_2 - 648va_2^2\sqrt{4(-9a_1^2 + 36a_0 a_2)^3 + (54a_1^3 - 324a_0 a_1 a_2 - 648va_2^2)^2}\right)^{1/3}} - \frac{\left(54a_1^3 - 324a_0 a_1 a_2 - 648va_2^2\sqrt{4(-9a_1^2 + 36a_0 a_2)^3 + (54a_1^3 - 324a_0 a_1 a_2 - 648va_2^2)^2}\right)^{1/3}}{6\sqrt[3]{2}\,a_2}. \tag{15}$$

Assuming polynomial form of $k(T)$, the desired approximate solution of the two-point boundary value problem (1) is (15) with the substitution $v(x) = v_i(x)$ for $1 \leq i \leq n$, provided that $v_i(x)$ is the ith approximate solution of transformed two point boundary value problem (3) with the boundary conditions $\gamma = F(\alpha)$ and $\mu = F(\beta)$.

3.2. $k(T)$ exponential function case

Choosing the thermal conductivity function $k(T)$ as the following exponential function

$$k(T) = a_0 \mathrm{e}^{a_1 T} \tag{16}$$

where the coefficients are positive constants, then by (2) and assuming $\rho = 0$

$$v = F(T) = \int_0^T a_0 \mathrm{e}^{a_1 s}\mathrm{d}s = \frac{a_0}{a_1}(\mathrm{e}^{a_1 T} - 1)$$

is the transformation F whose inverse F^{-1} corresponds to a function of v

$$T = \frac{1}{a_1}\ln\left(\frac{a_1}{a_0}v + 1\right). \tag{17}$$

The approximate solution of the two point boundary value problem (1) having the exponential form of $k(T)$ is (17) with the substitution $v(x) = v_i(x)$ for $1 \leq i \leq n$, provided that $v_i(x)$ is the ith approximate solution of transformed two point boundary value problem (3) with the boundary conditions $\gamma = F(\alpha)$ and $\mu = F(\beta)$.

3.3. $k(T)$ trigonometrical function case

Taking the thermal conductivity function $k(T)$ as the following trigonometric function

$$k(T) = \sin(\sigma T) + \cos(\sigma T) \tag{18}$$

where σ is a constant provided that $0 < \sigma \leq 1$, then by (2) and assuming $\rho = 0$

$$v = F(T) = \int_0^T (\sin(\sigma s) + \cos(\sigma s))\, \mathrm{d}s = \frac{1}{\sigma}(1 - \cos(\sigma T) + \sin(\sigma T))$$

is the transformation F whose inverse F^{-1} corresponds to a function of v

$$T = \frac{1}{\sigma} \arcsin \frac{1}{2}\left(-1 + \sigma v + \sqrt{1 + 2\sigma v - \sigma^2 v^2}\right). \tag{19}$$

The approximate solution of the two-point boundary value problem (1) having the trigonometric form of $k(T)$ is (19) with the substitution $v(x) = v_i(x)$ for $1 \le i \le n$, provided that $v_i(x)$ is the ith approximate solution of the transformed two point boundary value problem (3), and the boundary conditions are $\gamma = F(\alpha)$ and $\mu = F(\beta)$.

4. Numerical results

Some specific examples of the problem (1) are given in this section, which show the exactness and usefulness of the procedure in Sections 2 and 3. The first two examples are of the polynomial type, third one is of the exponential type and finally forth one is of the trigonometric type.

Problem 1. As a first example, the problem (1) in the interval (0, 1) under the homogeneous Dirichlet boundary conditions $T(0) = T(1) = 0$ is solved approximately in [17] by using the mentioned methodology. The function $k(T)$ is of the form (14) where the coefficients are $a_2 = 3, a_1 = 2, a_0 = 1$. It is easily seen that if we take $I = [0, \tau]$ where τ is an arbitrary positive constant, then $1 \le k(T) \le 3\tau^2 + 2\tau + 1$ in I. Taking $f(x, T)$ as

$$f(x, T) = \pi^2[-9(T^3 + T^2 + T) - 5\cos^2(\pi x) + 14\sin(\pi x) + 7],$$

the exact solution of this problem is given by $T(x) = \sin(\pi x)$. The approximate solution for this problem as functional approximation that is closed to the exact solution, is calculated and the specific values of the function and the errors are shown in Table 1 in [17].

Problem 2. We consider the problem (1) with $k(T) = T^4$ in the interval (0, 1) under the non-homogeneous Dirichlet boundary conditions $T(0) = 0$ and $T(1) = 1$ where $f(x, T)$ is taken

$$f(x, T) = \pi^2\left(-\frac{5}{4}T^5 - \sin\left(\frac{\pi x}{2}\right)^3\right).$$

The functional approximate solution of the Problem 2 is calculated by means of variational iteration method in [17] and it is

$$\tilde{T}(x) = T_1(x) = 2^{-4/5}\left(\sin\left(\frac{5}{2}\pi x\right) + 10\sin\left(\frac{1}{2}\pi x\right) - 5\sin\left(\frac{3}{2}\pi x\right)\right)^{\frac{1}{5}}$$

which is coincide with the analytic solution in the interval [0, 1].

Problem 3. For the exponential type example, we consider the problem (1) with $k(T) = \mathrm{e}^T$ in the interval (0, 1) under the nonhomogeneous Dirichlet boundary conditions $T(0) = 0$ and $T(1) = -1 + \ln\left(0.5(\mathrm{e}^2 + 1)\right)$. The function $k(T)$ is of the form (16) where the coefficients are $a_1 = 1, a_0 = 1$. It is easily seen that if we take $I = [0, \tau]$ where τ is an arbitrary positive constant, then $1 \le k(T) \le \mathrm{e}^\tau$ in I. If $f(x, T)$ is taken as

$$f(x, T) = \mathrm{e}^T \tag{20}$$

then the exact solution of this problem is given by $T(x) = -x + \ln\left(0.5(\mathrm{e}^{2x} + 1)\right)$.

Assuming $\rho = 0$, we apply the transformation $F(T) = v$ to Problem 3 where $k(T) = \mathrm{e}^T$ and since

$$v = F(T) = \int_0^T \mathrm{e}^s \mathrm{d}s = \mathrm{e}^T - 1,$$

we transform the Problem 3 into the two point boundary value problem:

$$\begin{aligned} &v''(x) = v(x) + 1, \quad 0 \le x \le 1, \\ &\text{subject to boundary conditions } v(0) = 0 \quad \text{and} \quad v(1) = \cosh 1 - 1. \end{aligned} \tag{21}$$

The Problem 3 is transformed to a suitable problem to apply the variational iteration method. Now, in order to find the optimal value of λ we have the iteration formula

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\xi)\left(v_n''(\xi) - v_n(\xi) - 1\right) \mathrm{d}\xi$$

which yields the below conditions by making the formula stationary.

$$\delta v_n(x) : 1 - \lambda'(\xi)|_{\xi=x} = 0$$
$$\delta v_n'(\xi) : \lambda(\xi)|_{\xi=x} = 0$$
$$\delta v_n(\xi) : \lambda''(\xi)|_{\xi=x} - \lambda(\xi)|_{\xi=x} = 0$$

are obtained and it follows that

$$\lambda(\xi) = \sinh(\xi - x).$$

Then, the iteration formula obtained as

$$v_{n+1}(x) = v_n(x) + \int_0^x \sinh(\xi - x)\left(v_n''(\xi) - v_n(\xi) - 1\right) d\xi. \tag{22}$$

Taking initial approximation as $v_0(x) = ax + b$, first iteration of (22) yields

$$v_1(x) = a\sinh(x) + b\cosh(x) + \cosh(x) - 1.$$

From the boundary conditions $v(0) = 0$ and $v(1) = \cosh 1 - 1$, the constant coefficients are $a = b = 0$ and the numeric solution satisfied two-point boundary value problem (21) is

$$v_1(x) = \cosh(x) - 1.$$

Since the thermal conductivity function is $k(T) = e^T$ and transformation F is

$$v = F(T) = e^T - 1 \quad \text{and} \quad \forall T \in I,$$

the inverse F^{-1} corresponds to a function of v

$$T = \ln(v + 1). \tag{23}$$

Noticing the substitution $v(x) = v_1(x)$ in (23), the numerical solution of Problem 3 is

$$\tilde{T}(x) = \ln(\cosh x).$$

The solution $\tilde{T}(x)$, is also the exact solution of the Problem 3.

Problem 4. For the trigonometric type example, we consider the problem (1) with $k(T) = \sin T + \cos T$ under the boundary conditions $T(0) = \pi/4$ and $T(1) = \text{arc}\,\cos[(2 + \sqrt{46})/10]$. The function $k(T)$ is of the form (18) where the argument of the function is $\sigma = 1$ in (18) and $f(x, T)$ is taken as

$$f(x, T) = -\sin(1 - \cos T + \sin T) \tag{24}$$

Assuming $\rho = 0$, we apply the transformation $F(T) = v$ to Problem 4 where $k(T) = \cos(T)$ and since

$$v = F(T) = \int_0^T (\sin s + \cos s)ds = 1 - \cos T + \sin T,$$

we transform the Problem 4 into the two point boundary value problem:

$$v''(x) + \sin(v(x)) = 0, \quad 0 \le x \le 1,$$

subject to boundary conditions $v(0) = 1$ and $v(1) = 0.6$.

The Problem 4 is now transformed to a mathematical pendulum where it is assumed that there is no friction. Therefore, the variational iteration method is applicable to the problem. It is found the optimal value of and the correction functional for the pendulum problem by He in [8], so that $\lambda(\xi) = \sin(\xi - x)$ and the correction functional is

$$v_{n+1}(x) = v_n(x) + \int_0^x \sin(\xi - x)\left(v_n''(\xi) + v_n(\xi) - \frac{1}{6}(v_n(\xi))^3 + \frac{1}{120}(v_n(\xi))^5\right) d\xi. \tag{25}$$

The previous iteration formula is constructed where the function is written in the series expansion form $\sin v \approx v - \frac{1}{6}v^3 + \frac{1}{120}v^5$. Since the unknown argument α is calculated $\alpha = \frac{13}{24}\sqrt{3}$ by means of the residual form, then taking the initial approximation as $v_0(x) = \cos(\alpha x)$ the first iteration of (22) yields

$$v_1(x) = \cos(\alpha x) - \frac{5}{886}\cos(3\alpha x) + \frac{1}{40330}\cos(5\alpha x) + \frac{50191}{8933095}\cos(x)$$

and it is satisfied the boundary conditions $v(0) = 1$ and $v(1) = 0.6$. Because of $v(x) \cong v_1(x)$, the numeric solution of the two-point boundary value problem, Problem 4, is

$$\tilde{T} = \text{arc}\,\sin\left(\frac{-1 + v_1(x) + \sqrt{1 + 2v_1(x) - v_1^2(x)}}{2}\right),$$

the inverse transformation F is the function of v such that

$$T = \text{arc } \sin\left(\frac{-1 + v + \sqrt{1 + 2v - v^2}}{2}\right).$$

This solution is the approximate solution of the problem.

5. Conclusion

The variational iteration method gives the functional approximate solutions of the problems. Transforming the strongly nonlinear two point boundary value problems in to the workable two point boundary value problems for He's variational iteration method, the numerical solutions are found. The first and the second test problems presented also in [17], have the polynomial type of thermal conductivity, while the third problem has the exponential type of the thermal conductivity and finally the fourth problem has the trigonometric type of the thermal conductivity. All of the problems in Section 4 are solved by means of the presented process in Section 3 and their numerical solutions are closed to their exact solutions. However in the Problem 2 and the Problem 3, the approximate solutions coincide with the exact ones. Consequently, by this paper it is shown that the presented method, the variational iteration method, is also convenient to apply the transformed problems in order to obtain their approximate solutions.

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